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Concerning Rates of Convergence of Riemann Sums

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1. INTRODUCTION AND RESULTS

For a Riemann integrable function f on the closed interval [0, 1], we let I denote the Riemann integral of f over [0, 1] and consider the Riemann sums

$$R_n(f; a) = \frac{1}{n} \sum_{k=1}^{n} f((k - a)/n),$$

where $0 \leqslant a \leqslant 1$; we set $R_n(f; 1/2) = R_n(f)$. Some well-known results are included in the following theorem.

THEOREM 1. (a) *If f is Riemann integrable on* [0, 1], *then*

$$R_n(f; a) - I = o(1) \textit{ as } n \to \infty \textit{ for each } a, \qquad 0 \leqslant a \leqslant 1.$$

(b) *If f is a function of bounded variation on* [0, 1] *then*

$$R_n(f; a) - I = O(1/n) \textit{ as } n \to \infty \textit{ for each } a, \qquad 0 \leqslant a \leqslant 1.$$

(c) *If f is absolutely continuous on* [0, 1], *then* $R_n(f) - I = o(1/n)$ *as* $n \to \infty$.

(d) *If f is differentiable on* [0, 1] *and its derivative f' is of bounded variation on* [0, 1], *then* $| R_n(f) - I | \leqslant T(f')/8n^2$ *for all n, where $T(f')$ is the total variation of f' on* [0, 1].

Now (a) is clear. The proof of (b) is easy and can be found, essentially, in [2]. Proofs of the weaker and somewhat different forms of (c) and (d) are given in [2]. A proof of (c) is given in [1], where it is also pointed out that the Riemann sums $R_n(f)$ cannot be replaced by $R_n(f; a)$ with $a \neq 1/2$. A proof of (d) is included in the following section, where we also indicate a unified proof of (b), (c) and (d) for $R_n(f) - I$ (i.e., for $a = 1/2$). We also show that $O(1/n^2)$ is the best possible estimate for $R_n(f) - I$ in (d), no matter how smooth the function f is. One of the main purposes of this paper is to construct examples showing also that the other estimates in Theorem 1 cannot be improved. Perhaps our methods of proof (particularly, the use of the saw-tooth functions in the proofs of Theorems 1, 4 and 5) are more interesting than our results, especially as some of these results may already be known.

THEOREM 2. *Let* $\{\epsilon_n\}$ *be a sequence of positive numbers which converges monotonically to zero. Then there is a Riemann integrable function* f *on* $[0, 1]$ *such that* $R_n(f; 0) - I \geqslant \epsilon_n$ *for all* n.

THEOREM 3. *There exist a positive number* ϵ_0, *an increasing function* f *of total variation less than one, and a sequence of positive integers* $n_k \to \infty$, *such that* $n_k\{I - R_{n_k}(f)\} \geqslant \epsilon_0$ *for all* k.

Theorem 3 shows that the hypothesis of absolute continuity in (c) of Theorem 1 cannot be replaced by that of bounded variation.

THEOREM 4. *Let* $\{\epsilon_n\}$ *be any sequence of positive numbers converging to zero. Then there is an absolutely continuous function* f *on* $[0, 1]$ *such that* $n_k\{I - R_{n_k}(f)\} \geqslant \epsilon_{n_k}$ *for* $k = 1, 2, \ldots$, *where* $\{n_k\}$ *is some sequence of positive integers tending to infinity.*

THEOREM 5. *If* f *is twice differentiable and* f'' *is bounded and almost everywhere continuous on* $[0, 1]$, *then*

$$\lim_{n\to\infty} n\{I - R_n(f)\} = \frac{1}{24}\int_0^1 f'' = \{f'(1) - f'(0)\}/24.$$

A proof of Theorem 5, assuming continuity of f'' on $[0, 1]$, is given in [2]. Our proof of Theorem 5 here will be a continuation of the proof of Theorem 1 (d). As a consequence of Theorem 5, we see that the rate of convergence of $R_n(f)$ to I cannot be improved to $o(1/n^2)$ unless $f'(0) = f'(1)$, no matter how smooth the function f is. In particular, if f'' is nonnegative and does not vanish on a set of positive measure, then $O(1/n^2)$ is the best estimate for $R_n(f) - I$.

COROLLARY. *If* f *is twice differentiable,* f'' *is bounded and almost everywhere continuous, and* $f'(0) = f'(1)$, *then* $R_n(f) - I = o(1/n^2)$.

2. The Saw-Tooth Functions

For $0 \leqslant s \leqslant 1$, we denote by χ_s the characteristic function of the closed interval $[s, 1]$, and for each positive integer n, we let

$$s_n = \sum_{k=1}^{n} \chi_{(k-1/2)/n} \, .$$

Consider the "saw-tooth" functions $v_n(t) = s_n(t) - nt$. For each n, $v_n(0) = v_n(1) = 0$, and v_n lies between $-1/2$ and $1/2$ and is linear with the exception of n unit jumps at the points $(k - 1/2)/n$, $k = 1,..., n$. It is also clear that if f is Riemann integrable on [0, 1], then

$$R_n(f) - I = \frac{1}{n} \int_0^1 f(t)\, dv_n(t).$$

Hence, if f is of bounded variation on [0, 1], then

$$R_n(f) - I = -\frac{1}{n} \int_0^1 v_n(t)\, df(t),$$

so that

$$| R_n(f) - I | \leqslant \frac{1}{2n} \int_0^1 | df(t)|.$$

That is, we have the following corollary of statement (b) of Theorem 1.

Corollary. *If f is a function of bounded variation on* [0, 1], *then* $| R_n(f) - I | \leqslant T(f)/2n$, *where $T(f)$ is the total variation of f on* [0, 1].

If f is absolutely continuous on [0, 1], then f' is Lebesgue integrable there and

$$R_n(f) - I = -\frac{1}{n} \int_0^1 f' v_n \, ,$$

so that by a proof similar to that of the Riemann–Lebesgue theorem, we see that $R_n(f) - I = o(1/n)$.

Now, let f be differentiable with f' of bounded variation on [0, 1]. Let $T(f')$ denote the total variation of f' on [0, 1], and let

$$u_n(x) = \int_0^x v_n(t)\, dt.$$

Then $u_n(k/n) = u_n(0) = 0$ for $k = 1,..., n$, each u_n is periodic with period $1/n$, and

$$\max_{0 \leqslant t \leqslant 1} | u_n(t)| = \int_0^{1/2n} - v_n(t)\, dt = 1/8n.$$

But

$$R_n(f) - I = -\frac{1}{n}\int_0^1 v_n f' = \frac{1}{n}\int_0^1 u_n \, df'.$$

Hence,

$$| R_n(f) - I | \leqslant T(f')/8n^2.$$

This completes the proof of statement (d) of Theorem 1.

Now, suppose that f'' is bounded and almost everywhere continuous on [0, 1], i.e., f'' is Riemann integrable there. Then

$$\begin{aligned} n^2\{R_n(f) - I\} &= n \int_0^1 u_n f'' \\ &= n \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} u_n f'' \\ &= n \sum_{k=1}^{n} \int_0^{1/n} u_n \left(t + \frac{k-1}{n}\right) f'' \left(t + \frac{k-1}{n}\right) dt \\ &= \int_0^{1/n} n u_n(t) \sum_{k=1}^{n} f'' \left(t + \frac{k-1}{n}\right) dt \\ &= \int_0^1 u_n(x/n) \sum_{k=1}^{n} f''((x + k - 1)/n) \, dx. \end{aligned}$$

Here, it can be seen that

$$\begin{aligned} u_n(x/n) &= \int_0^{x/n} v_n \\ &= \begin{cases} \displaystyle\int_0^{x/n} - nt \, dt & \text{if } \ 0 \leqslant x < 1/2, \\ \displaystyle -1/8n + \int_{1/2n}^{x/n} (1 - nt) \, dt & \text{if } \ 1/2 \leqslant x \leqslant 1, \end{cases} \\ &= \begin{cases} -x^2/2n & \text{if } \ 0 \leqslant x < 1/2, \\ -(1 - x)^2/2n & \text{if } \ 1/2 \leqslant x \leqslant 1. \end{cases} \end{aligned}$$

Let

$$w(x) = \begin{cases} x^2/2 & \text{if } \ 0 \leqslant x < 1/2, \\ (1 - x)^2/2 & \text{if } \ 1/2 \leqslant x \leqslant 1. \end{cases}$$

Then $\int_0^1 w = 1/24$ and $u_n(x/n) = -w(x)/n$ for all n and for $0 \leqslant x \leqslant 1$. Hence,

$$-n^2\{R_n(f) - I\} = \int_0^1 w(x) \, R_n(f''; 1 - x) \, dx,$$

which gives

$$\lim_{n\to\infty} n^2\{I - R_n(f)\} = \int_0^1 w \int_0^1 f'' = \{f'(1) - f'(0)\}/24.$$

This completes the proof of Theorem 5.

It is interesting to note the similarities between the rate of convergence of the sequence $R_n(f) - I$ to zero and that of the sequence of the Fourier coefficients $a_n(f)$ of the function f to zero. Indeed, consider both

$$R_n(f) - I = \frac{1}{n}\int_0^1 f(t)\, dv_n(t),$$

$$a_n(f) = \frac{i}{2\pi n}\int_0^1 f(t)\, de^{-i2\pi nt}.$$

The saw-tooth functions $v_n(t)$ in the study of convergence of $R_n(f)$ to the integral of f on [0, 1] play similar roles to those of the functions $e^{-i2\pi nt}$ in the study of Fourier series. The difficulty in working with the functions $v_n(t)$ is that they are *not* orthogonal. However, they still satisfy some interesting properties which are perhaps applicable to some approximation problems. In the following lemma we establish such a property, an application of which will be used in the proof of Theorem 4.

LEMMA. *Let m and n be positive integers such that the quotient m/n is an odd integer. Then*

$$\int_0^1 v_n v_m = \frac{n}{12m}\,.$$

In particular,

$$\int_0^1 v_n^2 = 1/12$$

for all positive integers n.

Proof. Write $m = (2k + 1)n$, $k \geqslant 0$. Since v_n has period $1/n$ and n divides m, $v_n v_m$ also has period $1/n$. Furthermore, on $[0, 1/n]$, $v_n v_m$ is symmetric about $1/2n$, and on $[0, 1/2n]$, $v_n(t) = -nt$.

Hence,

$$\begin{aligned}
\int_0^1 v_n v_m &= n \int_0^{1/n} v_n v_m = 2n \int_0^{1/2n} v_n v_m \\
&= -2n^2 \int_0^{1/2n} t v_m(t)\, dt \\
&= -2n^2 \int_0^{1/2n} t(s_m(t) - mt)\, dt \\
&= 2mn^2 \int_0^{1/2n} t\, dt - 2n^2 \sum_{j=1}^{k} j \int_{(2j-1)/2m}^{(2j+1)/2m} t\, dt \\
&= \frac{m}{12n} - \frac{2n^2}{m^2} \sum_{j=1}^{k} j^2 \\
&= n/12m.
\end{aligned}$$

3. Functions for Which the Estimates are Best

We shall now construct examples showing that the estimates in (a), (b), and (c) of Theorem 1 cannot be improved.

Proof of Theorem 2. This proof is easy. We define $f(x) = 0$ if $x = 0$ or if x is any irrational number in (0, 1). If x is a rational number in (0, 1], say $x = p/q$, where p and q are relatively prime positive integers, we define $f(x) = f(p/q) = \epsilon_q$. Since $\epsilon_q \to 0$, f is clearly continuous at each of the irrationals on [0, 1]. Hence, f is Riemann integrable on [0, 1]. Now,

$$\begin{aligned}
R_n(f; 0) - \int_0^1 f &= \frac{1}{n} \sum_{k=1}^{n} f(k/n) \\
&= \frac{1}{n} \sum_{k=1}^{n} \epsilon_{q_{k,n}} \geqslant \epsilon_n ,
\end{aligned}$$

where $k/n = p_{k,n}/q_{k,n}$, $(p_{k,n}, q_{k,n}) = 1$, so that $q_{k,n} \leqslant n$, $\epsilon_n \leqslant \epsilon_{q_{k,n}}$, for each $k = 1,..., n$, and $n = 1, 2,...$, by the monotonicity of the sequence $\{\epsilon_n\}$.

Proof of Theorem 3. Let p be a positive integer, chosen so large that

$$\epsilon_0 = \frac{1}{3p}(1 - 1/2^p) - 1/2^p > 0.$$

Order the rationals on [0, 1] as a sequence $\{x_j\}, j = 1, 2,...,$ so that $x_j = 1/3j$ for $j = 1,..., p$. Define the step functions

$$g_j(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant x_j, \\ 1/2^j & \text{if } x_j < x \leqslant 1, \end{cases}$$

$j = 1, 2,...$. We then define our function f by

$$f(x) = \sum_{j=1}^{\infty} g_j(x).$$

It is clear that this series converges uniformly on [0, 1]. Hence, f is an increasing function on [0, 1] with total variation $f(1) - f(0) < 1$.

Let $\nu_n(x)$ denote the number of points $(k - 1/2)/n$, $k = 1,..., n$, that lie in the interval $[0, x]$, and let $n_k = 3 \cdot (p + k)! - 1$, $k = 1, 2,...$. As usual, let $[t]$ be the integral part of t. Then for all $x \in [0, 1]$ and all $n = 1, 2,...,$ it is clear that

$$\nu_n(x) = [nx + 1/2].$$

Hence,

$$| \nu_n(x)/n - x | \leqslant 1/n;$$

and, for $x = x_j$, $j = 1,..., p$, $n = n_k$, $k = 1, 2,...,$ we also have

$$\begin{aligned} \nu_n(x_j) - nx_j &= \left[\frac{3 \cdot (p + k)! - 1}{3j} + \frac{1}{2}\right] - \frac{3 \cdot (p + k)! - 1}{3j} \\ &\geqslant \left[\frac{3 \cdot (p + k)!}{3j}\right] - \frac{3 \cdot (p + k)! - 1}{3j} \\ &= 1/3j \geqslant 1/3p. \end{aligned}$$

Therefore, for all $n = n_k$, $k = 1, 2,...,$

$$\begin{aligned} \int_0^1 f - R_n(f) &= \sum_{j=1}^{\infty} \left\{\int_0^1 g_j - \frac{1}{n} \sum_{k=1}^{n} g_j((k - 1/2)/n)\right\} \\ &= \sum_{j=1}^{\infty} \left\{(1 - x_j)/2^j - \left(1 - \frac{1}{n} \nu_n(x_j)\right)\Big/2^j\right\} \\ &= \sum_{j=1}^{p} \{\nu_n(x_j)/n - x_j\}/2^j + \sum_{j=p+1}^{\infty} \{\nu_n(x_j)/n - x_j\}/2^j \\ &\geqslant \frac{1}{n}\left\{\frac{1}{3p}(1 - 1/2^p) - 1/2^p\right\} \\ &= \epsilon_0/n. \end{aligned}$$

That is, $n_k\{I - R_{n_k}(f)\} \geqslant \epsilon_0 > 0$ for $k = 1, 2,...$.

Proof of Theorem 4. Choose a sequence of positive integers n_k, $0 < n_1 < n_2 < \cdots$, so that whenever $j < p$, n_p/n_j is an odd integer, and so that $12\epsilon_{n_k} \leqslant 1/2^k$ for $k = 1, 2,\ldots$. Let

$$g(t) = \sum_{j=1}^{\infty} v_{n_j}(t)/2^j,$$

where v_k are the saw-tooth functions defined in Section 2. Since $|v_k(t)| \leqslant 1/2$ for all k and all t in $[0, 1]$, the series converges uniformly to g. Hence, g is integrable on $[0, 1]$ and

$$\int_0^1 g = \sum_{j=1}^{\infty} \left\{\int_0^1 v_{n_j}\right\}/2^j = 0.$$

We now define our absolutely continuous function f by

$$f(x) = \int_0^x g.$$

It is clear that

$$\int_0^1 f = \int_0^1 (1-t)\, g(t)\, dt = -\int_0^1 tg(t)\, dt.$$

Hence,

$$\begin{aligned} n\left\{\int_0^1 f - R_n(f)\right\} &= -n\int_0^1 tg(t)\, dt - \sum_{k=1}^{n} \int_0^{(k-1/2)/n} g \\ &= -n\int_0^1 tg(t)\, dt + \sum_{k=1}^{n} \int_{(k-1/2)/n}^1 g \\ &= -n\int_0^1 tg(t)\, dt + \sum_{k=1}^{n} \int_0^1 \chi_{(k-1/2)/n}\, g \\ &= \int_0^1 g(t)\left\{\sum_{k=1}^{n} \chi_{(k-1/2)/n}(t) - nt\right\} dt \\ &= \int_0^1 g v_n \\ &= \sum_{j=1}^{\infty} \left\{\int_0^1 v_{n_j} v_n\right\}/2^j. \end{aligned}$$

Therefore, if $n = n_k$, $k = 1, 2,\ldots$, we can apply the lemma of Section 2 to obtain

$$n_k\{I - R_{n_k}(f)\} = \frac{1}{12}\left\{\frac{n_1}{2n_k} + \cdots + \frac{1}{2^k} + n_k/2^{k+1}n_{k+1} + \cdots\right\}$$
$$> 1/12 \cdot 2^k \geqslant \epsilon_{n_k}.$$

This completes the proof of the theorem.

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